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Stability analysis of $x = \pm y$ periodic orbits in a $\frac{1}{2}(Q - xy)^2$ potential

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Abstract. A combination of analytic and numerical methods is used to show the existence of simple stable orbits in the $\frac{1}{2}(Q - xy)^2$ potential.

1. Introduction

The spatial confinement of charged particles in spatially non-uniform magnetic fields forms the physical basis for most plasma fusion devices. However, the seemingly simple problem of the motion of even a single charged particle in a non-uniform magnetic field has turned out to be surprisingly complicated; most particle orbits turn out to be chaotic. Thus it is desirable to study in depth some simple magnetic configurations so as to obtain a better understanding of the physics of particle confinement. One such system is the two-dimensional cusp. This is the magnetic field generated by four straight line currents symmetrically situated at a large distance from the centre, which are parallel to the z axis of a three-dimensional rectangular coordinate system. The magnetic field can be defined through a vector potential $\boldsymbol{A}$ of the form $\boldsymbol{A} = xy\boldsymbol{k}$ where $\boldsymbol{k}$ is a unit vector along the z axis. The motion of a charged particle moving in such a magnetic field is governed by the Hamiltonian (see Rusbridge 1971)

$$H(x, y, p_x, p_y) = \tfrac{1}{2}\left(p_x^2 + p_y^2 + (Q - xy)^2\right) \tag{1}$$

where Q is a positive constant proportional to the particle momentum in the z direction.

This problem has been discussed by a number of authors. In particular, Rusbridge (1971, 1977) obtained numerical results for $Q < 1$ which nowadays would be classified as chaotic. Our own numerical results presented later confirm this. The presence of chaos leads to particles acquiring large values of x and y which physically correspond to the particle making large deviations from the central regions of the magnetic field, and this leads to particle loss from a practical containment device. For $Q \gg 1$ an adiabatic invariant exists which, if treated as an absolute constant of motion, gives conditions for absolute particle confinement. However, as Q approaches unity non-adiabatic effects appear which also lead to particle loss. These non-adiabatic effects were first discussed by Howard (1971) and in more detail by Cohen *et al* (1978) (subsequently to be referred to as CRF). A detailed analysis of the particle loss quantified in terms of enhanced diffusion is given in Cohen and Rowlands (1981).

It is worth noting that the special case $Q = 0$, where we have the potential x^2y^2, has many other physical applications (Dahlqvist and Russberg 1990, and references therein). It has been investigated extensively in order to test the belief that the motion is fully ergodic,

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which implies that no stable orbits should exist. However, Dahlqvist and Russberg have recently shown, using numerical simulations, the existence of at least one stable orbit, and so for $Q = 0$ the system is not fully ergodic (Dahlqvist and Russberg 1990). Our own numerical simulations show that the periodic orbit continues to exist for finite Q but only for $Q < 0.00002$.

However, with the simple constraint $x = \pm y$, it is easily shown that periodic solutions exist, and in fact their forms can be studied analytically. We will call such solutions the $x = \pm y$ orbits. These orbits can be expressed in terms of Jacobi elliptic functions. The main purpose of this paper is to show that for certain ranges of the parameter Q these orbits are stable.

Linearizing around the $x = \pm y$ orbits gives rise to two distinct stability equations. One equation is a form of Lamé's equation with solutions which are periodic and therefore stable. The other stability equation falls into two categories depending on the Q value. For all values of Q, except $Q = 1$, it is a type of Hill's equation, namely a homogeneous, linear, second-order differential equation with real periodic coefficients. With the usual form of Hill's equation, transition of the solution from stable to unstable behaviour is studied by varying a parameter (usually the eigenvalue). In contrast, in the present study, the period of the periodic term is also changed as the parameter, namely Q, is allowed to change. Nevertheless, the study of Hill's equation (Magnus and Winkler 1966) provides simple but practical tools to trace the stability transition which are readily extended to include our equations. They are the so-called Hill's discriminant and the oscillation theorem. Furthermore, the study by Churchill *et al* (1980) (subsequently to be referred to as CPR) ensures that our solutions will change their stability many times as the change in period of the periodic term is made, that is as Q is varied.

We have traced the stability regions of the $x = \pm y$ orbits numerically using the Hill's discriminant and find a complicated pattern, as a function of Q, for the existence of stable solutions. In particular, as $Q \to 1$, the regions of stability, which are interspersed with unstable ones, become smaller and smaller. In fact, because of the error build-up, the numerical calculations only show the existence of the first four or five of what turns out to be an infinite sequence of stable and unstable regions. The full nature of this complicated structure is revealed by a novel perturbation analysis valid as $Q \to 1$ which is discussed in section 5. Though this work was motivated by a problem related to particle confinement in fusion devices, the stability of periodic orbits in classical mechanics is important to other problems, for example, the relation between classical and quantum mechanical descriptions of a system (see, for example, Edmonds 1989).

2. Analytic expressions for the $x = \pm y$ periodic orbits

From the Hamiltonian (1) we may derive the equations of motion for a charged particle moving in the cusp magnetic field configuration

$$\begin{aligned} \frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= y(Q - xy) \\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= x(Q - xy) \end{aligned} \tag{2}$$

along with the energy conservation equation

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + (Q - xy)^2 = E. \tag{3}$$

The scaling property of the Hamiltonian system enables us to study this system with the normalized energy ($E = 1$).

For the sake of convenience, we use the linear transformation

$$p = \frac{x+y}{2} \qquad q = \frac{x-y}{2} \tag{4}$$

to obtain the equations of motion in the form

$$\begin{aligned} \frac{d^2p}{dt^2} &= +p(Q - p^2 + q^2) \\ \frac{d^2q}{dt^2} &= -q(Q - p^2 + q^2). \end{aligned} \tag{5}$$

It is easy to see that the $x = y$ orbits correspond to the case $q = 0$ and the $x = -y$ orbits to the case $p = 0$.

First, we consider the $x = y$ orbits labelled by $\bar{p}(t)$, for these periodic orbits the equations of motion reduce to

$$\left(\frac{d\bar{p}}{dt}\right)^2 = \frac{1}{2}\left((1 - Q^2) + 2Q\bar{p}^2 - \bar{p}^4\right). \tag{6}$$

This differential equation can be integrated in terms of elliptic functions and the solutions can be written in the following form:

When $Q \leqslant 1$

$$\bar{p}(t) = \sqrt{Q+1}\,\mathrm{cn}(t + t_0, k^2) \tag{7}$$

where k^2 ($= (Q+1)/2$) is the modulus of the Jacobi elliptic function. The Jacobi elliptic function is a doubly periodic function with period of $4\mathrm{K}(k^2)$ and $2\mathrm{K}(k^2) + 2\mathrm{iK}(1 - k^2)$, with K being the complete elliptic integral of the first kind.

When $Q \geqslant 1$

$$\bar{p}(t) = \sqrt{Q+1}\,\mathrm{dn}\left(\frac{\sqrt{Q+1}}{2}(t + t_0), k^2\right) \tag{8}$$

where $k^2 = 2/(1 + Q)$ with period $2\mathrm{K}(k^2)$. The Jacobi elliptic function, $\mathrm{dn}(t)$, has the property that it is always positive.

What is the significance of the point $Q = 1$? We see from (3) with $E = 1$ that since the kinetic energy must be greater or equal to zero, all orbits must satisfy $(Q - xy)^2 \leqslant 1$. This condition allows one to bound the regions in the x, y plane where orbits can exist. Then, since the boundaries are given by $y = (Q \pm 1)/x$, we see that when $Q > 1$, the confinement zone is split into two regions. The charged particle cannot jump between these regions. As Q decreases, these regions come together and join at $Q = 1$. For $Q < 1$, there is just one allowable region. Mathematically, the topology of the potential $(Q - p^2)^2$ when $0 < Q < 1$ and when $Q > 1$, is different. In general, the potential $(Q - p^2)^2$ can have two minima. These minima are connected when $0 < Q < 1$, but are not when $Q > 1$. In addition, the

$x = y$ orbit at $Q = 1$ forms a separatrix in the $(x, \mathrm{d}x/\mathrm{d}t)$ phase space. Typical orbits for three different values of Q are shown in figure 1.

Next, we consider the $x = -y$ orbits labelled by $\bar{q}(t)$. For these orbits, the equations of motion reduce to

$$\left(\frac{\mathrm{d}\bar{q}}{\mathrm{d}t}\right)^2 = \frac{1}{2}\left((1 - Q^2) - 2Q\bar{q}^2 - \bar{q}^4\right). \tag{9}$$

Unlike the $x = y$ case, the $x = -y$ orbits do not exist for $Q > 1$ because the right-hand side of (9) is then always negative. Integrating (9), we obtain the solution

$$\bar{q}(t) = \sqrt{1 - Q}\,\mathrm{cn}(t + t_0, k^2) \tag{10}$$

where $k^2 = \frac{1}{2}(1 - Q)$ with period $4\mathrm{K}(k^2)$ and $4\mathrm{iK}(1 - k^2)$.

As Q approaches 1, the amplitude of the oscillation goes to zero, which means the particle stays there forever, and so we have a trivial fixed point.

3. Stability equations

By linearizing around specific $x = y$ or $x = -y$ orbits ($\bar{q} = 0$ or $\bar{p} = 0$), the stability property of these periodic orbits can be studied.

First, we linearize about the $x = y$ orbits. That is, we write $p(t) = \bar{p}(t) + \delta p(t)$ and $q(t) = 0 + \delta q(t)$, substitute into (5) and neglect all products of δp and δq. The linear stability equations for $\delta p(t)$ are then, for the case $Q < 1$

$$\begin{aligned} &\frac{\mathrm{d}^2\delta p}{\mathrm{d}t^2} + (1 + 4k^2 - 6k^2\,\mathrm{sn}^2(t, k^2))\delta p = 0 \\ &\frac{\mathrm{d}^2\delta q}{\mathrm{d}t^2} + (-1 + 2k^2\,\mathrm{sn}^2(t, k^2))\delta q = 0 \end{aligned} \tag{11}$$

where $k^2 = (Q + 1)/2$, and for the case $Q > 1$

$$\begin{aligned} &\frac{\mathrm{d}^2\delta p}{\mathrm{d}s^2} + (4 + k^2 - 6k^2\,\mathrm{sn}^2(s, k^2))\delta p = 0 \\ &\frac{\mathrm{d}^2\delta q}{\mathrm{d}s^2} + (-k^2 + 2k^2\,\mathrm{sn}^2(s, k^2))\delta q = 0 \end{aligned} \tag{12}$$

where $s = \frac{1}{2}\sqrt{Q + 1}\,t$ and $k^2 = 2/(1 + Q)$.

For the special case $Q = 1$

$$\begin{aligned} &\frac{\mathrm{d}^2\delta p}{\mathrm{d}t^2} + (-1 + 6\,\mathrm{sech}^2(t))\delta p = 0 \\ &\frac{\mathrm{d}^2\delta q}{\mathrm{d}t^2} + (1 - 2\,\mathrm{sech}^2(t))\delta q = 0. \end{aligned} \tag{13}$$

All the above stability equations for $\delta p(t)$ are of the form of Lamé's equation

$$y'' + (\lambda - m(m + 1)k^2\,\mathrm{sn}^2(x, k^2))y = 0 \tag{14}$$

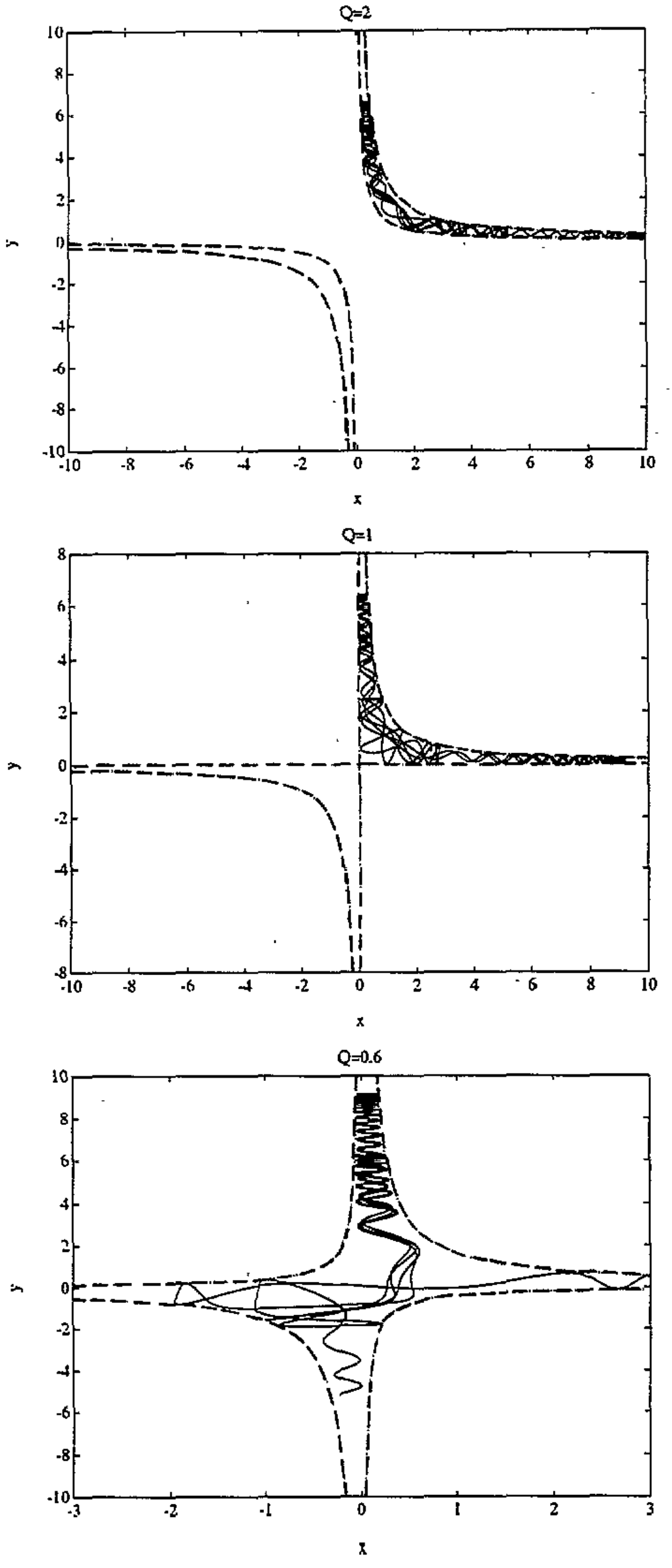

Figure 1. Typical orbits in the x–y plane for three different values of Q. The broken curves outline the confinement zone defined by $y = (Q \pm 1)/x$.

with $m = 2$ and $\lambda = 1 + 4k^2$ and $4 + k^2$. For this combination of parameters analytic solutions are known (Whittaker and Watson 1963). We have

$$\begin{aligned} \delta p(t) &= \mathrm{sn}(t, k^2)\,\mathrm{dn}(t, k^2) && \text{for } Q < 1 \\ \delta p(s) &= \mathrm{sn}(s, k^2)\,\mathrm{cn}(s, k^2) && \text{for } Q > 1 \\ \delta p(t) &= \tanh(t)\,\mathrm{sech}(t) && \text{for } Q = 1. \end{aligned} \tag{15}$$

It is easy to see that the solutions for the cases $Q < 1$ and $Q > 1$ are periodic, and so no instability arises in p-space. Similarly, even the solutions for $Q = 1$ are bounded, although not periodic. Hence, the $x = y$ orbit is stable in p-space for all values of Q. In fact since δp is seen to be proportional to $\mathrm{d}\bar{p}/\mathrm{d}t$, one can absorb δp into $\bar{p}$ as a simple phase factor.

The stability equations for δq are very close in form to Lamé's equation, but no integer values of m exist to fit these equations to the form of Lamé's equation as given by (14). According to the study by CPR and Magnus and Winkler (1966), stability equations of such form can have both stable and unstable solutions and have the following interesting properties:

(i) They have an infinite number of stability and instability intervals in k-space which implies Q-space, since k is an explicit function of Q.
(ii) The stability boundaries are determined by the equation $|\Delta(k^2)| = 2$, where $\Delta(k^2)$ is Hill's discriminant.
(iii) Let the period of $\mathrm{sn}^2(s, k^2)$ be $\mathrm{T}(k^2)$. If $\Delta(k^2) = 2$ at $k = k_1$, there exists a periodic solution of period $\mathrm{T}(k_1^2)$, and if $\Delta(k^2) = -2$ at $k = k_2$, there exists a periodic solution of period $2\mathrm{T}(k_2^2)$.
(iv) According to the oscillation theorem (Magnus and Winkler 1966), the stability property of a solution changes when passing through a stability boundary.

For our stability equations for δq, Hill's discriminant is defined by

$$\Delta(k^2) = \delta q_1(\mathrm{T}(k^2)) + \frac{\mathrm{d}\delta q_2}{\mathrm{d}t}(\mathrm{T}(k^2)) \tag{16}$$

where $\delta q_1(t)$ and $\delta q_2(t)$ are solutions of these stability equations with the following initial conditions: $\delta q_1(0) = (\mathrm{d}\delta q_2/\mathrm{d}t)(0) = 1$ and $\delta(\mathrm{d}q_1/\mathrm{d}t)(0) = \delta q_2(0) = 0$. This discriminant has to be computed numerically and the results are presented in the next section.

Next, we consider the stability of the $x = -y$ orbits, so now $p(t) = 0 + \delta p(t)$ and $q(t) = \bar{q}(t) + \delta q(t)$. Applying the same linearization method, we find

$$\begin{aligned} &\frac{\mathrm{d}^2\delta p}{\mathrm{d}t^2} + (-1 + 2k^2\,\mathrm{sn}^2(t, k^2))\delta p = 0 \\ &\frac{\mathrm{d}^2\delta q}{\mathrm{d}t^2} + (1 + 4k^2 - 6k^2\,\mathrm{sn}^2(t, k^2))\delta q = 0 \end{aligned} \tag{17}$$

where $k^2 = (1 - Q)/2$.

Comparison of these stability equations with those for the $x = y$ orbits with $Q < 1$, namely (11), shows that they are identical if δp and δq are interchanged. However, there are two differences that are worth mentioning and are associated with the different definitions of k^2. First, let us compare the $x = y$ periodic orbits governed by (7), $k^2 = (1 + Q)/2$, with the $x = -y$ periodic orbits of (10), $k^2 = (1 - Q)/2$. The amplitudes of $x = -y$ orbits become smaller when Q increases and they disappear at $Q = 1$. Next, let us compare the modulus of the Jacobi elliptic function. For the $x = y$ orbits it linearly increases from 0.5 to 1 when Q increases from 0 to 1, but for the $x = -y$ orbits it linearly decreases from 0.5 to 0.

4. Numerical results

Some of orbits governed by the expressions (7), (8) and (10) were numerically generated by using built-in elliptic functions in Mathematica. These sample orbits were compared to those generated by numerically integrating the equations of motion (2) with the $x = y$ and $x = -y$ constraints. They were found to be in very good agreement. During the test, we selected two values of Q from both ranges, $0 < Q < 1$ and $Q > 1$ for the $x = y$ orbits and one value of Q from $0 < Q < 1$ for the $x = -y$ orbits. This agreement shows that the elliptic functions we generate numerically and which are to be used in the subsequent stability analysis will not introduce significant errors.

Next, we computed Hill's discriminant to see its behaviour on a large scale from $Q = -1$ to $Q = 10$ and the results are shown in figure 2.

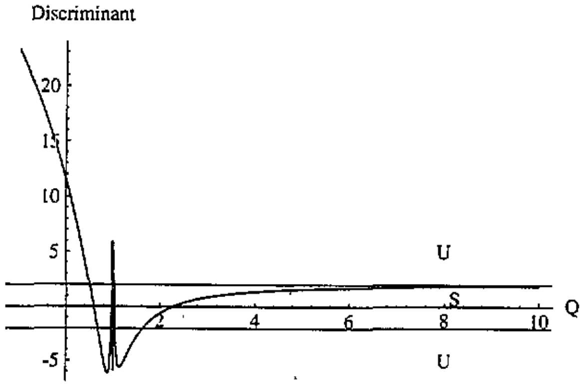

Figure 2. The broad features of the variation of Hill's discriminant $\Delta(k^2)$ shown as a function of Q. The horizontal lines at $\Delta = \pm 2$ separate the stable and unstable solutions. The U symbol denotes the unstable regions and the S symbol denotes the stable region.

The intersections of the discriminant curve with the $y = \pm 2$ lines indicate positions where stability transitions occur. According to the oscillation theorem, if one state of stability is known, then the rest are theoretically determined. Sohos *et al* (1989) show that the $x = y$ orbit for $Q = 0$ is unstable, and hence the solution of our stability equations will be stable if its discriminant is between -2 and 2, otherwise the solution will be unstable.

Let us focus on each region separately. For the $x = -y$ orbits, $-1 \leqslant Q < 0$, the result shows that the discriminant never passes the $+2$ horizontal line. Therefore, these orbits are always unstable. On the other hand, for the $x = y$ orbits there exist a number of transitions. When $0 \leqslant Q < 1$, the results show a number of stability transitions and suggest that this number is infinite since the discriminant oscillates faster and faster as Q approaches to 1. Also, there exists a recurrent structure as finer scale calculations are carried out. When $Q > 1$, it is found that the discriminant oscillates faster as Q decreases to 1, the same behaviour as for $Q < 1$ but in the other direction (see figure 3). Table 1 lists some of the values of Q where the discriminant cuts the ± 2 horizontal lines. These Q values will be called boundaries of stability.

A value of Q, namely $Q = 0.6$, for which the oscillation theorem predicts a stable solution was selected and the full dynamical equations, as given by (2), were numerically integrated. From these results a Poincaré surface of section was constructed by finding the values of x and dx/dt at the time when the orbits pierce through the $y = 0$ plane. This is

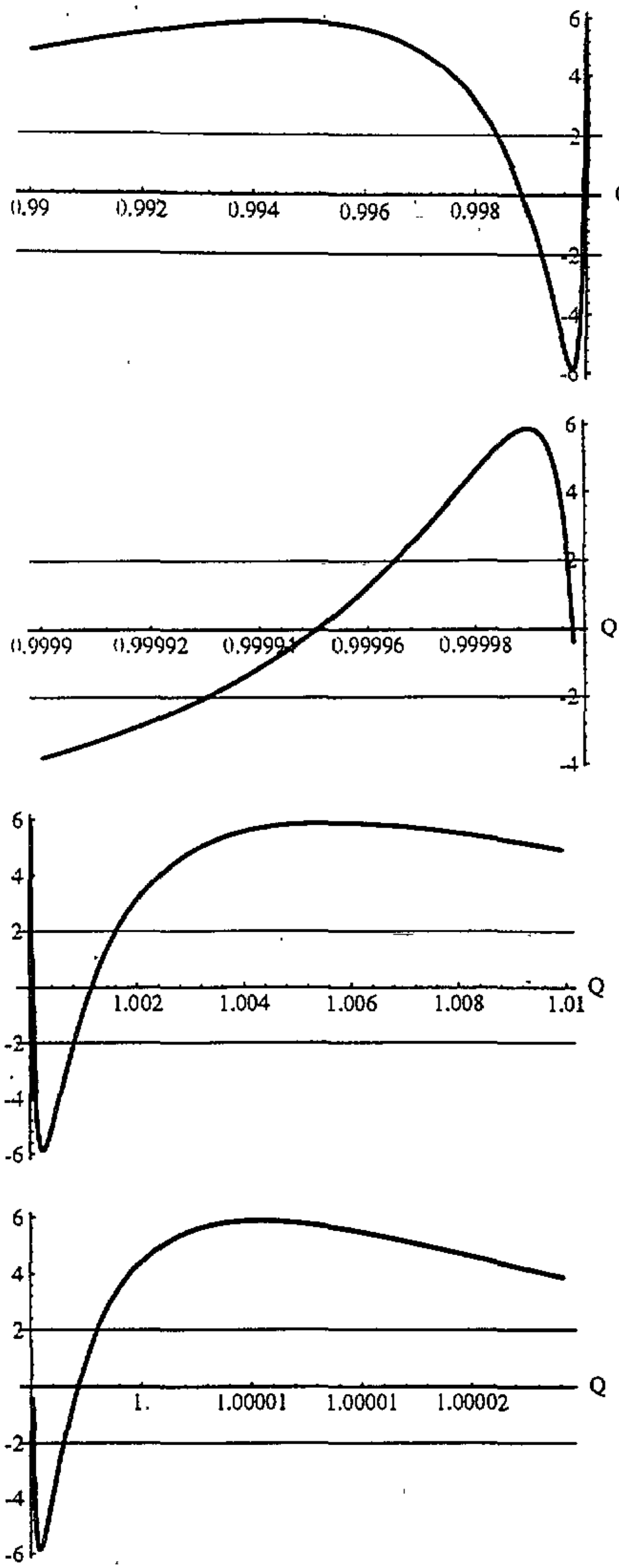

Figure 3. The fine scale variation of $\Delta(k^2)$ with Q.

Table 1. The values of Q together with the corresponding values of the Hill's discriminant $\Delta(k(Q)^2)$, very close to the various boundaries of stability as obtained numerically.

Q	$\Delta(k(Q)^2)$	Comments
0.506 37	+2.0000	1st boundary (stable)
0.673 40	−1.9997	2nd boundary (unstable)
0.964 24	−2.0000	3rd boundary (stable)
0.981 67	+2.0001	4th boundary (unstable)
0.998 38	+1.9998	5th boundary (stable)
1.000 81	−2.0000	5th boundary (stable)
1.001 62	+2.0000	4th boundary (unstable)
1.019 14	+1.9977	3rd boundary (stable)
1.039 44	−1.9999	2nd boundary (unstable)
1.635 88	−2.0001	1st boundary (stable)

shown in figure 4 and, as expected, a series of stable islands around the $x = y$ orbit are seen to exist. A point of interest is to study how these islands deform as the parameter Q is changed. First, as we move towards the first stability boundary by decreasing Q, the islands become more distorted and elongated. For values of Q less than the first boundary value ($Q \approx 0.506$) the islands break into two distinct families of stable islands. These two families of islands are not connected and thus, though we have a bifurcation, it is not period doubling. Around these stable islands, there is an expanding distorted figure-of-eight-shaped stochastic sea (see figure 5). This behaviour is first noticeable at $Q = 0.48$. This stochastic sea becomes thicker as we decrease Q further and finally covers almost the whole of phase space at $Q = 0$.

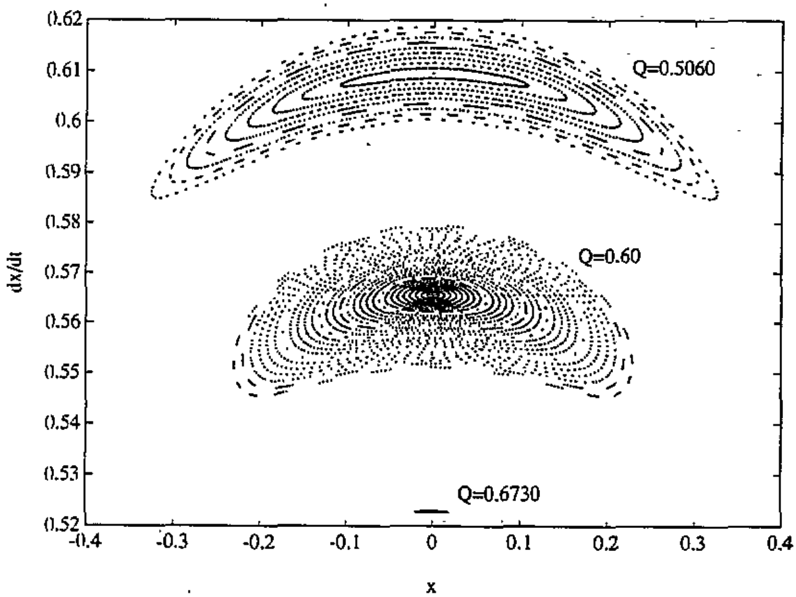

Figure 4. The change in the form of the stable islands that exist as Q changes from 0.506 to 0.673. The Poincaré surface of section is the $y = 0$ plane.

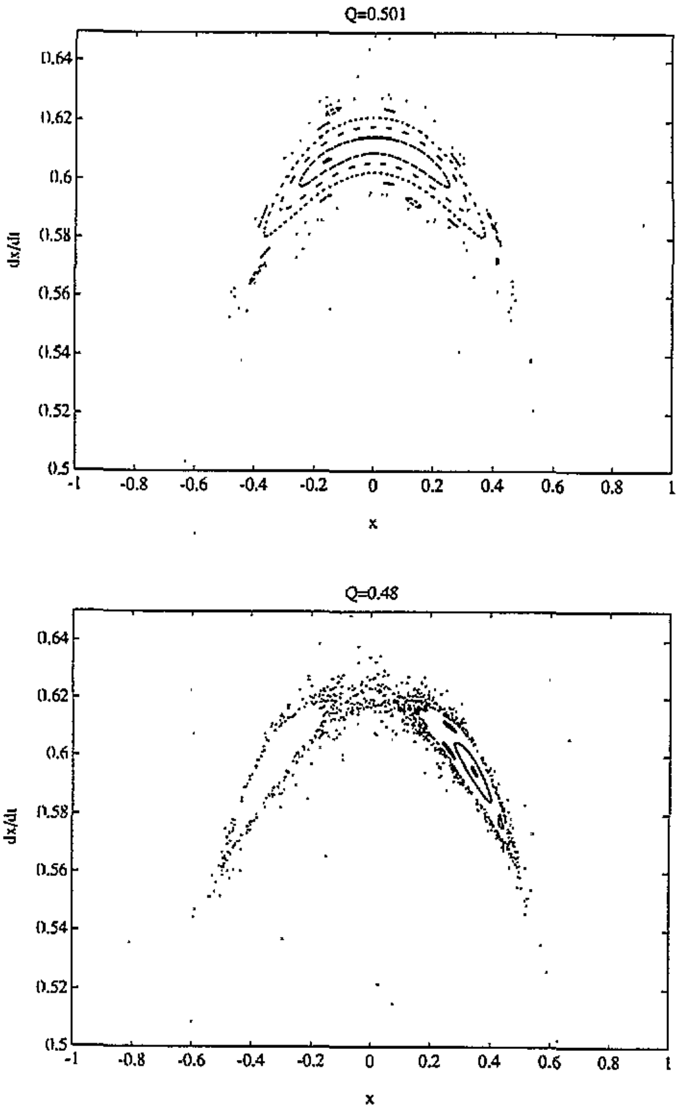

Figure 5. The formation of new periodic orbits and the growth of a chaotic sea about the weakly unstable $x = y$ orbits.

Importantly, these numerical simulations show the existence of yet another set of stable orbits which have bifurcated from the unstable $x = y$ orbit.

Next, we changed Q so as to move towards the second stability boundary, that is by increasing Q. The stable islands get smaller and smaller, finally disappearing at the second boundary $Q \approx 0.673$ (see figure 4). For $0.673 < Q < 0.964$, the orbits are unstable and the phase plane looks chaotic.

As we move towards the third boundary at $Q = 0.964$, we find a stochastic sea surrounding two families of stable islands, and these two separate islands start to connect and at the same time the stochastic sea gets thinner and finally disappears at the boundary. This is qualitatively the same phenomenon that occurs as one approaches the first boundary at $Q \approx 0.506$ from smaller values of Q. For Q values beyond the third boundary, we simply have stable $x = y$ solutions surrounded by stable islands until the value of Q reaches that at the fourth boundary. Hence, we assume that this phenomenon occurs at every odd boundary. Nevertheless, as the rate of oscillation of the discriminant increases, it becomes more and more difficult to obtain numerically the values of Q defining the boundaries.

For $Q > 1$, we observe a similar behaviour as we decrease Q to 1. The stable islands around the $x = y$ orbit split. However, these split islands are now connected and so we have a period doubling bifurcation. This is illustrated in figure 6.

Finally, we consider the special case $Q = 1$. Because the period of the periodic coefficient of Hill's equation becomes infinite, the discriminant becomes undefined and it is impossible to reach the point numerically; a special treatment is needed. We tried to

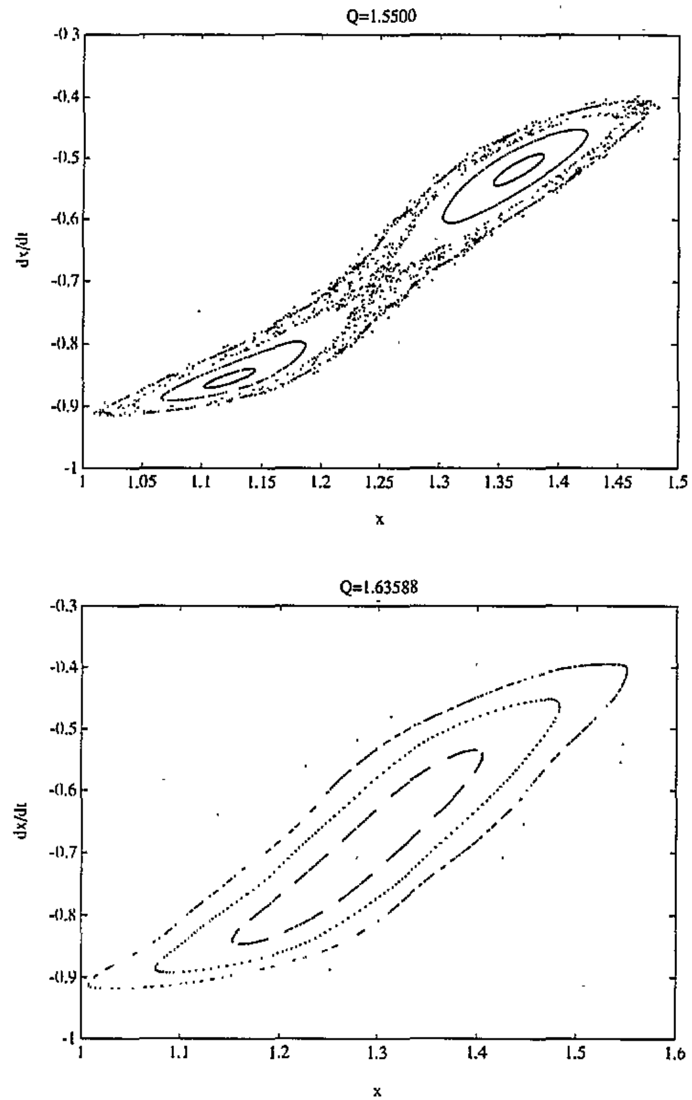

Figure 6. A period doubling bifurcation of the $x = y$ periodic orbit at the first boundary of $Q > 1$. The Poincaré surface of section is selected at the $y = \sqrt{Q} - x$ plane.

produce a Poincaré surface of section but failed due to the build-up of the numerical errors. At $Q \equiv 1$, the $x = y$ orbit forms a separatrix and its period is infinity. If a small variation is added to the initial conditions of this orbit, it will soon grow because the trajectory crosses the separatrix to the other side. This conclusion is confirmed by the fact that the trajectory is deformed significantly when we change the step of integration. If we focus on the stability equation for this case, we see that for large enough time the solution is approximately simple harmonic, because the coefficient $(1 - 2\,\mathrm{sech}^2(t))$ tends to 1 very quickly. As a result, we conclude that the $x = y$ orbit for $Q \equiv 1$ is stable.

5. Analytic treatment

The numerical results discussed in the last section suggest that as $Q \to 1$ the regions of stability become narrower and narrower and their number increase. Due to build-up of numerical error associated with the fact that the period of the orbits approach infinity as $Q \to 1$, it becomes impossible to follow this break up of stable regions after the first few. Nevertheless, the complicated behaviour around $Q = 1$ can be understood analytically by making a suitable approximation to the periodic coefficients in the linear stability equations. The approximation is based on a representation of elliptic functions as infinite series of pulses (solitons). For example (Toda 1981),

$$\mathrm{dn}^2(\xi) = \left(\frac{\mathrm{E}}{\mathrm{K}} - \frac{\pi}{2\mathrm{K}\mathrm{K}'}\right) + \left(\frac{\pi}{2\mathrm{K}'}\right)^2 \sum_{l=-\infty}^{+\infty} \mathrm{sech}^2\left[\frac{\pi}{2\mathrm{K}'}(\xi - 2l\mathrm{K})\right] \tag{18}$$

where K is the complete elliptic integral of the first kind, E is the complete elliptic integral of the second kind, and K$'$ is $\mathrm{K}(1 - k^2 = k'^2)$.

For $Q \to 1$, we have $k^2 \to 1$ and $k'^2 \to 0$. Using well known expansions for the elliptic integrals in the limit as $k^2 \to 1$ (see Byrd and Friedman 1954), the linear stability equations (11) and (12) for δq both become, to lowest significant order

$$\frac{\mathrm{d}^2\delta q}{\mathrm{d}t^2} + \left(1 - 2\sum_{l=-\infty}^{+\infty} \mathrm{sech}^2(t - 2l\mathrm{L})\right)\delta q = 0 \tag{19}$$

where $\mathrm{L} = \ln(4/\sqrt{1-k^2})$.

The above stability equation immediately reminds one of a quantum problem where a plane wave propagates through a periodic potential which can, for large enough values of L, be thought of as an infinite number of isolated but equivalent 'valleys'. These valleys being periodically located at $t = 2l\mathrm{L}$. Therefore, $\delta q(t)$ behaves like a plane wave for a substantial range of t, before it encounters a particular valley which is described by $-2\,\mathrm{sech}^2(t)$. The solution to the stability problem can then be analysed in terms of a wave scattering through the potential valleys located at $2l\mathrm{L}$ on the t axis.

Before a scattering we may write the solution in the form $\delta q(t) = A\mathrm{e}^{\mathrm{i}t} + B\mathrm{e}^{-\mathrm{i}t}$ and after the scattering in the form $\delta q(t) = C\mathrm{e}^{\mathrm{i}t} + D\mathrm{e}^{-\mathrm{i}t}$, where A, B, C and D are complex constants. These constants are related to each other by the following equations

$$
\begin{aligned}
C &= A\left(\tau - \frac{\rho^2}{\tau}\right) + \frac{\rho}{\tau}B \\
D &= \frac{B}{\tau} - \frac{\rho}{\tau}A
\end{aligned}
\tag{20}
$$

where $|\rho|^2 = R$, the reflection coefficient, and $|\tau|^2 = T$, the transmission coefficient. These coefficients are given in Morse and Feshbach (1953).

By selecting initial conditions according to the definition of the discriminant, we can analytically compute this quantity and find

$$
\Delta(Q) = 2p_0 \cos(2\mathrm{K}(Q) - \theta) \tag{21}
$$

where

$$
p = \frac{1}{\tau} = \frac{\Gamma(1-\mathrm{i})\Gamma(-\mathrm{i})}{\Gamma\left(\frac{1}{2} + (\sqrt{7/4} - 1)\mathrm{i}\right)\Gamma\left(\frac{1}{2} - (\sqrt{7/4} + 1)\mathrm{i}\right)}
$$

so that $p_0 = \mathrm{Abs}(p) \approx 2.938\,763$ and $\theta = \mathrm{Arg}(p) \approx 2.384\,130$.

The values of Q at the stability boundaries $\Delta(Q) = \pm 2$ are then easily found and are given by

$$
Q_n = \begin{cases}
\left[2\left(1 - 16\exp\{-\theta - (-1)^n \sin^{-1}(C_\pm) - \left(n - \frac{1}{2}\right)\pi\}\right) - 1\right] & \text{for } Q < 1 \quad (22) \\[2ex]
\left[\dfrac{2}{\left(1 - 16\exp\{-\theta - (-1)^n \sin^{-1}(C_\pm) - \left(n - \frac{1}{2}\right)\pi\}\right)} - 1\right] & \text{for } Q > 1 \quad (23)
\end{cases}
$$

where $C_\pm$ is $\pm 1/p_0$ for $\Delta = \pm 2$.

Table 2. The values of Q, as obtained using (22) and (23), together with the corresponding values of the Hill's discriminant $\Delta(k(Q)^2)$. The column labelled % gives the percentage deviation of these values of Q from those obtained by the direct numerical solution of (11) and (12) for $\delta q(t)$ as given in table 1.

Q	%	$\Delta(Q)$	Comments
0.132 358 36	−70.14	+2	1st boundary (stable)
0.566 734 29	−15.84	−2	2nd boundary (unstable)
0.962 505 81	−0.1801	−2	3rd boundary (stable)
0.981 276 89	−0.0398	+2	4th boundary (unstable)
0.998 379 73	-2.7×10^{-5}	+2	5th boundary (stable)
1.000 809 43	-6.0×10^{-5}	−2	5th boundary (stable)
1.001 621 58	0.000 16	+2	4th boundary (unstable)
1.018 900 04	−0.0236	+2	3rd boundary (stable)
1.038 210 53	−0.1185	−2	2nd boundary (unstable)
1.553 081 28	−5.1	−2	1st boundary (stable)

The above approximate treatment is good for values of Q near unity. It gives agreement with the values of Q obtained numerically to within 6%, as given in table 1 except for the first and second boundaries for $Q < 1$ (see table 2). The increase in the rate of oscillation of the discriminant as we move towards $Q = 1$ is also described by (21) and it is readily

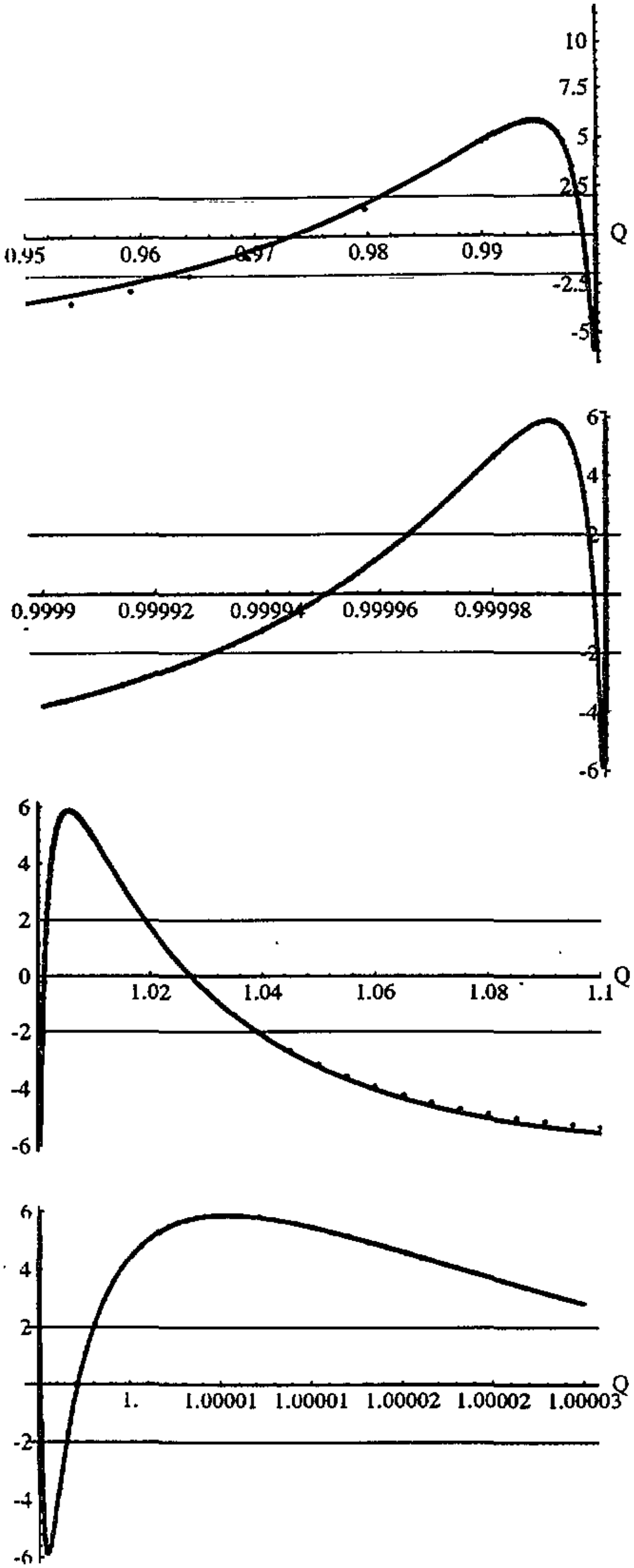

Figure 7. In these figures dots represent the discriminant computed by the numerical integration of the linear stability equations and curves represent the discriminant generated by (21). As we move toward $Q = 1$, a better approximation is obtained.

seen that the system undergoes an infinite number of stability transitions. The method also gives agreement with the numerically generated values of the discriminant within this range of Q. In figure 7 the value of the discriminant as a function of Q is shown. The dots corresponding to the value calculated from a numerical solution of (11) and (12). Thus, this analysis complements the numerical treatment.

In addition, it is found that the ratio of the amplitudes of the harmonic wave, before and after scattering through a valley, yields the map

$$\psi_{n+1} = \frac{(\psi_n - \rho)}{\left[(\tau^2 - \rho^2) + \rho\psi_n\right]} e^{-4iK}. \tag{24}$$

This map can be brought into the form of a tangent map

$$X_{n+1} = \eta - 1/X_n \tag{25}$$

where $\eta = \left((\tau^2 - \rho^2)e^{2iK} + e^{-2iK}\right)/\tau$ and $X = e^{-2iK}\left((\tau^2 - \rho^2) + \rho\psi\right)/\tau$, and this equation can be solved exactly (see Rowlands 1990). Thus an analytic form for the solution $\delta q(t)$ can be obtained by this method.

6. Conclusion

We have studied the $x = \pm y$ orbits for the $(Q - xy)^2$ potential and have generated the orbits analytically. Their stability has been examined in detail. We are able to trace their stability as a function of Q by combining numerical and analytic methods. A very rich and intricate structure has been revealed, and, importantly, windows of stability have been found. The full equations of motion, (2), have been solved numerically for values of Q in the neighbourhood of these stable windows. In particular, the changes of structure in a Poincaré surface of section from closed orbits to chaotic ones has been revealed. These changes are illustrated in figures 4 and 5 when the surface of section is selected at the $y = 0$ plane. These same sections reveal the existence of further periodic orbits for values of Q for which the $x = y$ orbits are unstable.

For $Q \gtrsim 1.636$, only the $x = y$ orbit exists and this is stable for all larger values of Q. The existence of such a stable periodic orbit is intimately linked with the changes in the adiabatic invariant μ, since from both the analytic and numerical studies (CRF and Howard 1971), it is known that the major change in μ occurs in the median plane which is just the $x = y$ orbit. This connection is under further study.

Particles injected into the periodic orbits or the stable islands will of course remain in them indefinitely. It is expected that particles whose orbits pass near these periodic ones will tend to stay in their vicinity. Such attachment has been identified in studies of dynamical systems governed by maps (Yannacopoulos and Rowlands 1993, and references therein) and shown to reduce the effective diffusion of particles through phase space. A study of the effects of stable regions in what is otherwise a chaotic phase space is an important general problem. The model potential discussed in this paper particularly for $Q < 1$ has the desirable added feature of simplicity. Further study of the quantitative effects of such periodic orbits is underway.

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